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MAXWELL'S EQUATIONS IN SPHERICALLY SYMMETRIC MEDIA

by

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ABSTRACT

This report shows how to express the components of any electromagnetic field in a medium for which $\mathcal{E}$ and μ are functions only of r, the distance from the origin, in terms of two scalar functions, V and W. Differential equations for V and W are derived. These results are deduced in a special curvilinear coordinate system which yields formally simpler differential equations for these two functions and which makes possible the calculation of the components by simple differentiation. The material is potentially applicable to the problem of anomalous propagation over a spherical earth.

1. Introduction.

Present attempts to treat the problem of anomalous propagation of electromagnetic waves, that is, the problem of propagation over the earth in a medium of variable refractive index, make significant approximations. One treats the surface of the earth as flat, and partially correct for this assumption by changing the true law of variation of the index of refraction to what is known technically as the modified index of refraction. Another treats the spherical earth but neglects terms in differential equations so that the resulting procedure is equivalent to the use of a flat earth and the modified index. This latter procedure is employed by Furry, Pekeris and others[1]; the former, by Luneburg[2]. These approximate treatments also use the simplifying assumption that the index of refraction varies only with height above the earth.

A flat earth theory using a modified index of refraction, or its equivalent, is inadequate in several respects. As the frequency of the radiation increases the approximation becomes poorer. It becomes poorer too at greater heights above the surface of the earth. Aside from approximations introduced, the calculations which must be performed to obtain practical data are considerable and at present a major obstacle to the widespread use of these theories.

For these reasons a theory of anomalous propagation which would deal directly with a spherical earth and the true variation of the index of refraction with height above the surface seems very desirable. Of course one could not be sure that such a theory would give results more amenable to calculation than present ones.

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1. The theory employed by Furry and Pekeris is presented in Radiation Laboratory Report #680: Theory of Characteristic Functions in Problems of Anomalous Propagation, February 28, 1945.

   Calculations based on this theory have appeared in several articles in the Proc.I.R.E., Journal of Applied Physics, and British journals during the past few years.

2. Luneburg, R. K.: Propagation of Electromagnetic Waves from an Arbitrary Source Through Inhomogeneous Stratified Atmospheres, Research Report No. 172-6, submitted by New York University to Watson Laboratories, U. S. Air Forces, December, 1947.

Some preliminary work has been done towards developing the mathematics of a theory of propagation over a spherical earth in media stratified in the form of concentric spherical layers. Mathematically stated this type of stratification requires that the dielectric constant ϵ and the permeability μ are functions of r only where r is one of the spherical coordinates r, θ, φ. The statement that the ϵ and μ of the medium are functions of r only may also be made in the form that the medium possesses spherical symmetry. The variation of ϵ and μ with φ and θ is ignored because at present it is impossible to make any progress on this more complicated problem.

This paper reports on what might be described as the first stage of the work. It shows that if ϵ and μ are functions of r only, then three components of the electric field vector E and the magnetic field vector H can be determined readily from two scalar functions V and W. The differential equations which V and W must satisfy are also obtained. These results are embodied in formulas (5.9) and (5.10) of the paper.

It will be noted that the results are expressed in a somewhat uncommon curvilinear coordinate system rather than the more familiar spherical system. The advantages lie in somewhat simpler differential equations for V and W than are obtainable in spherical coordinates (compare equations (3.5) below) and the fact that it is possible to obtain the six field components from V and W by simple differentiations. The formulation of the problem of solving Maxwell's equations in this coordinate system also makes possible an interesting comparison between the use of rectangular coordinates with ϵ and μ as functions of z and the use of spherical coordinates with ϵ and μ as functions of r. This comparison has in fact been utilized to investigate some questions on the subject of the modified index of refraction.

The major work on the problem of anomalous propagation in spherically stratified media remains to be done. In particular one needs to determine V and W so as to satisfy the boundary conditions of the physical problem. Also, if experience is any guide, the problem of carrying out calculations with such a solution will be a major one. The present material was written up partly to show

concretely the course of one of our present investigations. Also it now seems likely that it will be utilized in a subsequent report on the modified index of refraction.

2. Basic Facts about Maxwell's Equations.

The determination of an electromagnetic field calls for solving Maxwell's equations

$$\begin{aligned} \text{curl } H - \frac{\epsilon}{c} E_t &= \frac{4\pi\sigma}{c} E \\ \text{curl } E - \frac{\mu}{c} H_t &= 0 \end{aligned} \tag{2.1}$$

for the vectors $E = (E_1, E_2, E_3)$ and $H = (H_1, H_2, H_3)$, where E_i and H_i are real functions of x,y,z and t.

Under the influence of a periodic source the fields E and H are periodic. This periodic time dependence can be decomposed into a sum of sine and cosine terms since any periodic function can be so expressed. Then E and H can be represented as

$$\begin{aligned} E &= \text{real part of } \bar{E}, \text{ where } \bar{E} = E^* e^{-i\omega t} \\ H &= \text{real part of } \bar{H}, \text{ where } \bar{H} = H^* e^{-i\omega t} . \end{aligned} \tag{2.2}$$

$E^* = e_1 + ie_2$ and $H^* = h_1 + ih_2$ are complex functions of x,y, and z.

We may require that the equations (2.1) hold for $\bar{E}$ and $\bar{H}$ for they will in that case hold for the real parts alone. That is, we shall treat the equations

$$\begin{aligned} \text{curl } \bar{H} - \frac{\epsilon}{c} \bar{E}_t &= \frac{4\pi\sigma}{c} \bar{E} \\ \text{curl } \bar{E} + \frac{\mu}{c} \bar{H}_t &= 0 \end{aligned} \tag{2.3}$$

For time periodic solutions of equations (2.1) it is possible to

reexpress equations (2.3) in the form

$$\text{a)}\quad \operatorname{curl} \bar{H} - \frac{\epsilon^*}{c} \bar{E}_t = 0$$
$$\text{b)}\quad \operatorname{curl} \bar{E} + \frac{\mu}{c} \bar{H}_t = 0 \qquad (2.4)$$

where $\epsilon^* = \epsilon$ if $\sigma = 0$ and $\epsilon^* = \epsilon + \frac{4\pi\sigma}{\omega} i$ when $\sigma \neq 0$.

At any point in space where there is no source of electromagnetic energy the vectors E and H satisfy the additional equations div $\epsilon E = 0$ and div $\mu H = 0$. We may as before replace E and H by $\bar{E}$ and $\bar{H}$ and add to equation (2.4) the equations

$$\text{c)}\quad \operatorname{div} \epsilon \bar{E} = 0$$
$$\text{d)}\quad \operatorname{div} \mu \bar{H} = 0. \qquad (2.5)$$

By using the fact that the divergence of the curl of any vector function is zero, it follows that equations (2.5) are consequences of equations (2.4) if div $\epsilon \bar{E}$ = div $\mu \bar{H} = 0$ at any particular time t. In the case of time periodic solutions the latter two equations are a direct consequence of the first two.

In working with equations (2.4) and (2.5) we shall assume $\sigma = 0$ and write ϵ then for ϵ^*. However the case $\sigma \neq 0$ can then be always be found by replacing ϵ by ϵ^*. We shall also write E for $\bar{E}$ and H for $\bar{H}$. However it is the complex E and H of (2.2) with which we shall be dealing except where explicitly noted otherwise.

By taking the curl of equation (2.4b) and substituting it in the time derivative of equation (2.4a) we obtain

$$\frac{\epsilon\mu}{c^2} E_{tt} + \mu \operatorname{curl} \frac{1}{\mu} \operatorname{curl} E = 0 \qquad (2.6a)$$

By reversing the roles of the two equations (2.4) we obtain

$$\frac{\epsilon\mu}{c^2} H_{tt} + \epsilon \operatorname{curl} \frac{1}{\epsilon} \operatorname{curl} H = 0. \qquad (2.6b)$$

With the aid of the vector identities[3]

$$\mu \operatorname{curl} \frac{1}{\mu} \operatorname{curl} E = -\Delta E + \operatorname{grad} \operatorname{div} E + \mu \left(\operatorname{grad} \frac{1}{\mu}\right) \times \operatorname{curl} E$$

and

$$\frac{1}{\epsilon} \operatorname{div} \epsilon E = 0 = \operatorname{div} E + \frac{1}{\epsilon} (\operatorname{grad} \epsilon) \cdot E \tag{2.7}$$

we can write equations (2.6) in the form

$$\begin{aligned} \frac{\epsilon\mu}{c^2} E_{tt} - \Delta E &= \operatorname{grad} (p \cdot E) + q \times \operatorname{curl} E \\ \frac{\epsilon\mu}{c^2} H_{tt} - \Delta H &= \operatorname{grad} (q \cdot H) + p \times \operatorname{curl} H \end{aligned} \tag{2.8}$$

where the vectors p and q are convenient abbreviations for

$$\begin{aligned} p &= \operatorname{grad} (\log \epsilon) = \frac{1}{\epsilon} \operatorname{grad} \epsilon = -\epsilon \operatorname{grad} \frac{1}{\epsilon} \\ q &= \operatorname{grad} (\log \mu) = \frac{1}{\mu} \operatorname{grad} \mu = -\mu \operatorname{grad} \frac{1}{\mu} \end{aligned} \tag{2.9}$$

If ϵ and μ are functions of three variables then the six scalar equations involved in (2.8) involve all components of E or H and it is impossible to solve for any one component readily. For a stratified medium, however, where ϵ and μ depend on one variable only it is possible in some coordinate systems to obtain one equation involving only one component of E and one equation involving only one component of H, which equations may be solvable. Moreover it is possible in these systems to obtain next the other components of E and H.

The treatment of equations (2.8) in rectangular coordinates with ϵ and μ as functions of z was given in another report[4]. In article 3 we shall show how these components of E and H may be isolated in the spherical coordinate system with ϵ and μ as functions of r.

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3. See Stratton, J.A.: Electromagnetic Theory, p. 604, relations (9) and (10). McGraw-Hill Book Co., N.Y., 1941.

4. See footnote 2.

3. Specialization of Maxwell's Equations to Media of Spherical Symmetry.

Since $\epsilon = \epsilon(r)$ and $\mu = \mu(r)$ we have, by application of the gradient in spherical coordinates, that

$$\begin{aligned} p &= \frac{\epsilon'}{\epsilon} \text{ grad } r \\ q &= \frac{\mu'}{\mu} \text{ grad } r. \end{aligned} \tag{3.1}$$

By denoting the radial components of E and H by E_o and H_o, that is,

$$\begin{aligned} E_o &= E \cdot \text{grad } r \\ H_o &= H \cdot \text{grad } r \end{aligned} \tag{3.2}$$

we may write the first of equations (2.7) as

$$\frac{\epsilon\mu}{c^2} E_{tt} - \Delta E = \text{grad } \left(\frac{\epsilon'}{\epsilon} E_o\right) + \frac{\mu'}{\mu} \text{grad } r \times \text{curl } E$$

and hence, taking the radial component,

$$\frac{\epsilon\mu}{c^2} E_{o_{tt}} - \text{grad } r \cdot \Delta E = \frac{\partial}{\partial r}\left(\frac{\epsilon'}{\epsilon} E_o\right) \tag{3.3}$$

Now by introducing the vector

$$\vec{r} = r \text{ grad } r = xi + yj + zk$$

we have[5]

$$\Delta(rE_o) = \Delta(\vec{r} \cdot E) = \vec{r} \cdot \Delta E + 2 \text{ div } E .$$

Furthermore, by utilizing the second relation (2.8) wherein we now use the fact that ϵ is a function of r only, we obtain

$$\frac{1}{\epsilon} \text{ div } \epsilon E = 0 = \text{div } E + \frac{\epsilon'}{\epsilon} E_o$$

---

5. The relation $\Delta(\vec{r} \cdot E) = \vec{r} \cdot \Delta E + 2 \text{ div } E$ is best established in rectangular coordinates by applying Δ in the form $\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} + \frac{\partial^2}{\partial z^2}$. By virtue of the invariant character of the scalar product the relationship must hold in spherical coordinates also, with the appropriate meaning of ΔE.

Hence from these last two relations we may write

$$\vec{r} \cdot \Delta E = r \operatorname{grad} r \cdot \Delta E = \Delta(rE_o) + \frac{2\epsilon'}{\epsilon} E_o$$

Use of the second equality in relation (3.3) gives

$$\frac{\epsilon\mu}{c^2} E_{o_{tt}} - \frac{1}{r} \Delta(rE_o) = \frac{2\epsilon'}{r\epsilon} E_o + \frac{\partial}{\partial r} \left(\frac{\epsilon'}{\epsilon} E_o\right) \qquad (3.4)$$

i.e. an equation for the radial component E_o. A similar consideration for the magnetic vector H leads to the result: The radial components E_o and H_o of E and H satisfy the equations

$$\frac{\epsilon\mu}{c^2} E_{o_{tt}} - \frac{1}{r} \Delta(rE_o) = 2 \frac{\epsilon'}{r\epsilon} E_o + \frac{\partial}{\partial r} \left(\frac{\epsilon'}{\epsilon} E_o\right) \qquad (3.5)$$

$$\frac{\epsilon\mu}{c^2} H_{o_{tt}} - \frac{1}{r} \Delta(rH_o) = 2 \frac{\mu'}{r\mu} H_o + \frac{\partial}{\partial r} \left(\frac{\mu'}{\mu} H_o\right)$$

We now have a differential equation for the single component E_o and one for H_o. We could proceed to seek their solutions and then show how to derive the other components of E and H. Such a course would lead to considerable complication. It is mathematically preferable to adopt a more indirect procedure. We shall first choose a new coordinate system and express Maxwell's equations in it. We shall then show that E and H can be derived from two scalar functions V and W and establish the differential equations which V and W must satisfy in this new coordinate system.

4. Some Vector Theory for a Special Curvilinear Coordinate System.

Let (ξ_1, ξ_2, ξ_3) be curvilinear coordinates[6] related to Cartesian rectilinear coordinates (X_1, X_2, X_3) by three functions

$$X_i = X_i(\xi_1, \xi_2, \xi_3) \quad , \quad i = 1,2,3 \qquad (4.1)$$

We introduce the three vectors

$$S_1 = \frac{\partial X_1}{\partial \xi_1} \cdot \frac{\partial X_2}{\partial \xi_1} \cdot \frac{\partial X_3}{\partial \xi_1}$$

$$S_2 = \frac{\partial X_1}{\partial \xi_2} \cdot \frac{\partial X_2}{\partial \xi_2} \cdot \frac{\partial X_3}{\partial \xi_3} \qquad (4.2)$$

$$S_3 = \frac{\partial X_1}{\partial \xi_3} \cdot \frac{\partial X_2}{\partial \xi_3} \cdot \frac{\partial X_3}{\partial \xi_3}$$

and assume that the coordinate surfaces $\xi_1 = \text{const.}$, $\xi_2 = \text{const.}$, $\xi_3 = \text{const.}$, are orthogonal. Since S_1, for example, is obtained by holding ξ_2 and ξ_3 constant it is normal to the family of surfaces $\xi_1 = \text{const.}$ Hence S_1, S_2, and S_3 are mutually orthogonal vectors, or

$$S_1 \cdot S_2 = S_2 \cdot S_3 = S_3 \cdot S_1 = 0. \qquad (4.3)$$

The three vectors S_1, S_2, S_3 are in general not unit vectors. Their lengths are given by three functions $A_i(\xi_1, \xi_2, \xi_3)$ such that[7],

$$S_1^2 = A_1 \; ; \; S_2^2 = A_2 \; ; \; S_3^2 = A_3 \qquad (4.4)$$

---

6. The discussion of the next few pages is merely the specialization to a particular coordinate system of standard material on curvilinear coordinates. See, for example, Stratton, loc. cit., pp. 38-59. The vectors S_1, S_2, S_3 here are the vectors a_1, a_2, a_3 of Stratton.

7. The A_i here are the h_i^2 of Stratton. See formula (70) of p. 48 there.

We also assume that the Jacobian $\frac{\partial(X_i)}{\partial(\xi_k)}$ of the transformation $X_i = X_i(\xi_1, \xi_2, \xi_3)$ is not zero so that we may invert: $\xi_i = \xi_i(X_1, X_2, X_3)$.

By virtue of the orthogonality of the vectors S_1, S_2, S_3 one proves[8] that

$$\text{grad } \xi_i = \frac{1}{A_i} S_i , \quad i = 1,2,3 \qquad (4.5)$$

and that the Jacobian $(\frac{\partial X_i}{\partial \xi_k})$ has the value[9]

$$(\frac{\partial X_i}{\partial \xi_k}) = \sqrt{A_1 A_2 A_3} \qquad (4.6)$$

Note also that the vectors $\frac{1}{\sqrt{A_i}} S_i$ form a system of orthogonal unit vectors so that

$$\frac{S_1 \times S_2}{\sqrt{A_1 A_2}} = \frac{S_3}{\sqrt{A_3}} ; \quad \frac{S_2 \times S_3}{\sqrt{A_2 A_3}} = \frac{S_1}{\sqrt{A_1}} ; \quad \frac{S_3 \times S_1}{\sqrt{A_3 A_1}} = \frac{S_2}{\sqrt{A_2}} .$$

A vector $\vec{U}$ in space is characterized by its components U_1, U_2, U_3 relative to the vectors S_1, S_2, S_3 so that

$$\vec{U} = U_1 S_1 + U_2 S_2 + U_3 S_3 \qquad (4.7)$$

Its divergence div $\vec{U}$, in terms of the generalized components, is given by the expression[10]

---

8. The general expression for gradient $\phi(\xi_1, \xi_2, \xi_3)$ in an orthogonal curvilinear coordinate system is $\nabla \phi = \sum_{\nu=1}^{3} \frac{\partial \varphi}{\partial \xi_i} \frac{S_i}{A_i}$. See formula (78) on p. 49 of Stratton. To obtain (4.5) let $\varphi = \xi_i$.

9. In orthogonal coordinates this Jacobian is $a_1 \cdot a_2 \times a_3$. This scalar triple product has the value $\sqrt{g} = h_1 h_2 h_3$ in Stratton's notation. See p. 43 and p. 48 there.

10. This is equivalent to formula (79) of Stratton.

$$\operatorname{div} \vec{U} = \sum_{\nu=1}^{3} \left(\frac{\partial U_\nu}{\partial \xi_\nu} + U_\nu \frac{\partial P}{\partial \xi_\nu} \right) \tag{4.8}$$

where $P = \log \sqrt{A_1 A_2 A_3}$.

For the curl of the vector one finds[11]

$$\operatorname{curl} \vec{U} = \sum_{\nu,\mu} \frac{S_\nu \times S_\mu}{\sqrt{A_\nu A_\mu}} \left\{ \sqrt{\frac{A_\mu}{A_\nu}} \frac{\partial U_\mu}{\partial \xi_\nu} - \sqrt{\frac{A_\nu}{A_\mu}} \frac{\partial U_\nu}{\partial \xi_\mu} + \frac{2}{\sqrt{A_\nu A_\mu}} \sum_{\alpha=1}^{3} U_\alpha \left(S_\mu \cdot \frac{\partial S_\nu}{\partial \xi_\alpha} \right) \right\} \tag{4.9}$$

where the summation refers to all combinations ν, μ with $\nu \neq \mu$, i.e., yields three terms: (1,2); (2,3); (3,1) . In other words the three components C_1, C_2, C_3 of the vector

$$\operatorname{curl} \vec{U} = C_1 S_1 + C_2 S_2 + C_3 S_3$$

are given by the expressions

$$C_1 = \frac{1}{\sqrt{A_1}} \left\{ \sqrt{\frac{A_3}{A_2}} \frac{\partial U_3}{\partial \xi_2} - \sqrt{\frac{A_2}{A_3}} \frac{\partial U_2}{\partial \xi_3} + \frac{2}{\sqrt{A_2 A_3}} \sum_{\alpha=1}^{3} U_\alpha \left(S_3 \cdot \frac{\partial S_2}{\partial \xi_\alpha} \right) \right\}$$

$$C_2 = \frac{1}{\sqrt{A_2}} \left\{ \sqrt{\frac{A_1}{A_3}} \frac{\partial U_1}{\partial \xi_3} - \sqrt{\frac{A_3}{A_1}} \frac{\partial U_3}{\partial \xi_1} + \frac{2}{\sqrt{A_1 A_3}} \sum_{\alpha=1}^{3} U_\alpha \left(S_1 \cdot \frac{\partial S_3}{\partial \xi_\alpha} \right) \right\} \tag{4.10}$$

$$C_3 = \frac{1}{\sqrt{A_3}} \left\{ \sqrt{\frac{A_2}{A_1}} \frac{\partial U_2}{\partial \xi_1} - \sqrt{\frac{A_1}{A_2}} \frac{\partial U_1}{\partial \xi_2} + \frac{2}{\sqrt{A_2 A_3}} \sum_{\alpha=1}^{3} U_\alpha \left(S_2 \cdot \frac{\partial S_1}{\partial \xi_\alpha} \right) \right\}$$

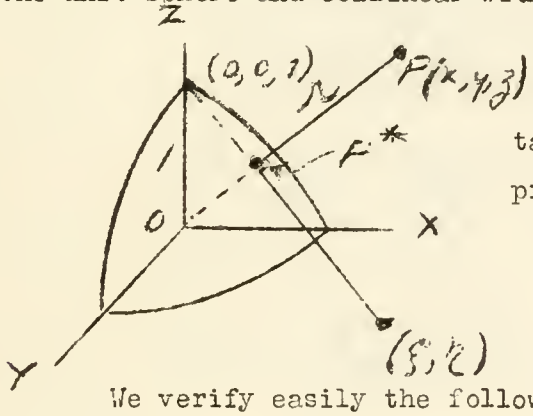
We apply the above formulae to a particular curvilinear coordinate system. We introduce as coordinates for a point P whose rectangular coordinates are x,y,z the quantities

$$\xi_1 = r, \quad \xi_2 = \xi, \quad \xi_3 = \eta$$

---

11. This is equivalent to formula (80) of Stratton.

Here r is the usual spherical coordinate. ξ , η are the rectangular coordinates of the point obtained by stereographic projection of P* onto the x,y-plane, where P* is on the unit sphere and collinear with O and P. The use of these coordinates ξ , η instead of the angular coordinates Θ , φ has certain formal advantages for our problem as we shall see presently.



We verify easily the following relations which hold under stereographic projection:

$$\xi = \frac{x}{r-z} \qquad x = \frac{2r\xi}{1+\rho^2}$$

$$\eta = \frac{y}{r-z} \qquad y = \frac{2r\eta}{1+\rho^2} \qquad (4.11)$$

$$\rho^2 = \xi^2 + \eta^2 = \frac{r+z}{r-z} \qquad z = \frac{\rho^2-1}{\rho^2+1}\, r$$

The relations of our coordinates ξ , η to the ordinary angular coordinates Θ, φ are given by the equations

$$\xi = \cot \Theta/2 \cos\varphi$$
$$\eta = \cot \Theta/2 \sin\varphi \qquad (4.12)$$

Indeed by introducing these expressions in relations (4.11) we find the familiar formula

$$x = r \sin\Theta\cos\varphi$$
$$y = r \sin\Theta\sin\varphi \qquad (4.13)$$
$$z = r \cos\Theta$$

From relations (4.11) we obtain the vectors

$$S_1 = (x_r, y_r, z_r)$$
$$S_2 = (x_\xi, y_\xi, z_\xi)$$
$$S_3 = (x_\eta, y_\eta, z_\eta)$$

namely,

$$S_1 = \left(\frac{2\xi}{\rho^2+1}, \frac{2\eta}{\rho^2+1}, \frac{\rho^2-1}{\rho^2+1}\right)$$

$$S_2 = \left(2r\frac{1+\eta^2-\xi^2}{(\rho^2+1)^2}, -\frac{4r\xi\eta}{(\rho^2+1)^2}, \frac{4r\xi}{(\rho^2+1)^2}\right) \qquad (4.14)$$

$$S_3 = \left(-\frac{4r\xi\eta}{(\rho^2+1)^2}, 2r\frac{1+\xi^2-\eta^2}{(\rho^2+1)^2}, \frac{4r\eta}{(\rho^2+1)^2}\right)$$

which satisfy the relation

$$S_1 \cdot S_2 = S_2 \cdot S_3 = S_3 \cdot S_1 = 0$$
$$S_1^2 = 1 \; ; \; S_2^2 = S_3^2 = A = \frac{4r^2}{(\rho^2+1)^2} \qquad (4.15)$$

5. Representation of E and H in Media of Spherical Symmetry by Two Scalar Functions.

We next express Maxwell's equations in the r, ξ, η coordinates. First let us write E and H as

$$E = \frac{1}{A}(E_1 S_1 + E_2 S_2 + E_3 S_3)$$
$$H = \frac{1}{A}(H_1 S_1 + H_2 S_2 + H_3 S_3) \qquad (5.1)$$

Note that since S_1 is a unit vector in the direction of r,

$$\frac{E_1}{A} = E_o \quad \text{and} \quad \frac{H_1}{A} = H_o$$

are the radial components of the vectors E and H.

By applying our general formula (4.10) for the curl of a vector to the vectors E and H as given by (5.1) we may write Maxwell's equations (2.4) in the r, ξ, η coordinate system thus:

$$\frac{\epsilon}{c}\dot{E}_1 = \frac{\partial H_3}{\partial \xi} - \frac{\partial H_2}{\partial \eta}\,, \quad -\frac{\mu}{c}\dot{H}_1 = \frac{\partial E_3}{\partial \xi} - \frac{\partial E_2}{\partial \eta}$$

$$\frac{\epsilon}{c}\dot{E}_2 = \frac{\partial}{\partial \eta}\left(\frac{H_1}{A}\right) - \frac{\partial H_3}{\partial r}\,, \quad -\frac{\mu}{c}\dot{H}_2 = \frac{\partial}{\partial \eta}\left(\frac{E_1}{A}\right) - \frac{\partial E_3}{\partial r} \qquad (5.2)$$

$$\frac{\epsilon}{c}\dot{E}_3 = \frac{\partial H_2}{\partial r} - \frac{\partial}{\partial \xi}\left(\frac{H_1}{A}\right)\,, \quad -\frac{\mu}{c}\dot{H}_3 = \frac{\partial E_2}{\partial r} - \frac{\partial}{\partial \xi}\left(\frac{E_1}{A}\right)$$

The expressions for div ϵE and div μH are, by virtue of formula (4.8):

$$\operatorname{div} \epsilon E = \frac{1}{A}\frac{\partial}{\partial r}(\epsilon E_1) + \frac{\epsilon}{A}\left(\frac{\partial E_2}{\partial \xi} + \frac{\partial E_3}{\partial \eta}\right) = 0$$

$$\operatorname{div} \mu H = \frac{1}{A}\frac{\partial}{\partial r}(\mu H_1) + \frac{\mu}{A}\left(\frac{\partial H_2}{\partial \xi} + \frac{\partial H_3}{\partial \eta}\right) = 0$$

From the first of equations (5.2) it follows that

$$\frac{\epsilon\mu}{c^2}E_{1_{tt}} = \frac{\mu}{c}\left(\frac{\partial \dot{H}_3}{\partial \xi} - \frac{\partial \dot{H}_2}{\partial \eta}\right)$$

Hence by using the expressions for $\dot{H}_3$ and $\dot{H}_2$ in (5.2), it follows that

$$\frac{\epsilon\mu}{c^2}E_{1_{tt}} = \left(\frac{\partial^2}{\partial \xi^2} + \frac{\partial^2}{\partial \eta^2}\right)\frac{E_1}{A} - \frac{\partial}{\partial r}\left(\frac{\partial E_2}{\partial \xi} + \frac{\partial E_3}{\partial \eta}\right)$$

By the first of equations (5.3) we have that

$$\frac{\partial E_2}{\partial \xi} + \frac{\partial E_3}{\partial \eta} = -\frac{1}{\epsilon}\frac{\partial}{\partial r}(\epsilon E_1)$$

Hence

$$\frac{\epsilon\mu}{c^2}\frac{\partial^2 E_1}{\partial t^2} = \left(\frac{\partial^2}{\partial \xi^2} + \frac{\partial^2}{\partial \eta^2}\right)\frac{E_1}{A} + \frac{\partial}{\partial r}\frac{1}{\epsilon}\left(\frac{\partial}{\partial r}(\epsilon E_1)\right)$$

and similarly: (5.4)

$$\frac{\epsilon\mu}{c^2}\frac{\partial^2 H_1}{\partial t^2} = \left(\frac{\partial^2}{\partial \xi^2} + \frac{\partial^2}{\partial \eta^2}\right)\frac{H_1}{A} + \frac{\partial}{\partial r}\frac{1}{\mu}\left(\frac{\partial}{\partial r}(\mu H_1)\right)$$

which are two second order equations for the components E_1 and H_1.

We now determine the remaining components E_2, E_3, H_2, H_3. From equations (5.2) and (5.3) we have immediately that

$$\frac{\partial E_2}{\partial \xi} + \frac{\partial E_3}{\partial \eta} = -\frac{1}{\epsilon}\frac{\partial}{\partial r}(\epsilon E_1) \quad ; \quad \frac{\partial H_2}{\partial \xi} + \frac{\partial H_3}{\partial \eta} = -\frac{1}{\mu}\frac{\partial}{\partial r}(\mu H_1)$$
$$\frac{\partial E_3}{\partial \xi} - \frac{\partial E_2}{\partial \eta} = -\frac{\mu}{c}\frac{\partial H_1}{\partial t} \quad ; \quad \frac{\partial H_3}{\partial \xi} - \frac{\partial H_2}{\partial \eta} = \frac{\epsilon}{c}\frac{\partial E_1}{\partial t} \quad . \qquad (5.5)$$

We now assume that ϵE_1 and μH_1 can be found from two scalar functions V and W by means of the relations

$$\epsilon E_1 = -\left(\frac{\partial^2 V}{\partial \xi^2} + \frac{\partial^2 V}{\partial \eta^2}\right)$$
$$\mu H_1 = -\left(\frac{\partial^2 W}{\partial \xi^2} + \frac{\partial^2 W}{\partial \eta^2}\right) \qquad (5.6)$$

Since there are any number of solutions for V and W to these differential equations there is no restriction on ϵE_1 and μH_1 imposed by requiring them to be obtained in this way.

By introducing these expressions in equation (5.4) we conclude that E_1 and H_1 satisfy the differential equations (5.4) <u>if</u> V and W satisfy the equations

$$\frac{\epsilon\mu}{c^2} V_{tt} = \frac{1}{A}(V_{\xi\xi} + V_{\eta\eta}) + \epsilon \frac{\partial}{\partial r}\left(\frac{1}{\epsilon} V_r\right)$$

$$\frac{\epsilon\mu}{c^2} W_{tt} = \frac{1}{A}(W_{\xi\xi} + W_{\eta\eta}) + \mu \frac{\partial}{\partial r}\left(\frac{1}{\mu} W_r\right) \qquad (5.7)$$

Expressions for the components E_2, H_2, E_3 and H_3 are then given by

$$\epsilon E_2 = V_{r\xi} - \frac{\epsilon}{c} W_{\eta t} \; ; \; \mu H_2 = W_{r\xi} + \frac{\mu}{c} W_{\eta t}$$

$$\epsilon E_3 = V_{r\eta} + \frac{\epsilon}{c} W_{\xi t} \; ; \; \mu H_3 = W_{r\eta} - \frac{\mu}{c} V_{\xi t} \qquad (5.8)$$

That these values of E_2, H_2, E_3 and H_3 satisfy Maxwell's equations (5.5) may be seen by substituting directly and using relations (5.6).

The results of this article may be summed up in the following principal theorem:

If V and W are two solutions of the partial differential equations

$$\frac{\epsilon\mu}{c^2} V_{tt} = \frac{1}{A}(V_{\xi\xi} + V_{\eta\eta}) + \epsilon \frac{\partial}{\partial r}\left(\frac{1}{\epsilon} V_r\right)$$

$$\frac{\epsilon\mu}{c^2} W_{tt} = \frac{1}{A}(W_{\xi\xi} + W_{\eta\eta}) + \mu \frac{\partial}{\partial r}\left(\frac{1}{\mu} W_r\right) \qquad (5.9)$$

then a solution of Maxwell's equations is obtained by the formulae:

$$\begin{aligned}
\epsilon E_1 &= -V_{\xi\xi} - V_{\eta\eta} \quad , \quad \mu H_1 = -W_{\xi\xi} - W_{\eta\eta} \\
\epsilon E_2 &= V_{r\xi} - \frac{\epsilon}{c} W_{\eta t} \quad , \quad \mu H_2 = W_{r\xi} + \frac{\mu}{c} V_{\eta t} \\
\epsilon E_3 &= V_{r\eta} + \frac{\epsilon}{c} W_{\xi t} \quad , \quad \mu H_3 = W_{r\eta} - \frac{\mu}{c} V_{\xi t}
\end{aligned} \qquad (5.10)$$

It is presupposed that ϵ and μ are functions of r only.

The meaning of this result is that the electromagnetic vectors E and H of expression (5.1) are given by the formulae:

$$\epsilon E = \frac{S_1}{A}\left(- V_{\xi\xi} - V_{\eta\eta} \right) + \frac{S_2}{A}\left(V_{r\xi} - \frac{\epsilon}{c} W_{\eta t}\right) + \frac{S_3}{A}\left(V_{r\eta} + \frac{\epsilon}{c} W_{\xi t}\right)$$

$$\mu H = \frac{S_1}{A}\left(- W_{\xi\xi} - W_{\eta\eta} \right) + \frac{S_2}{A}\left(W_{r\xi} + \frac{\mu}{c} V_{\eta t}\right) + \frac{S_3}{A}\left(W_{r\eta} - \frac{\mu}{c} V_{\xi t}\right) \qquad (5.11)$$

Further, the right sides are curl vectors[12], namely

$$\epsilon E = \text{curl } \mathcal{A}$$

$$\mu H = \text{curl } \mathcal{B} \qquad (5.12)$$

where

$$\mathcal{A} = - \frac{\epsilon}{c} W_t S_1 + \frac{1}{A}\left(V_\eta S_2 - V_\xi S_3\right)$$

$$\mathcal{B} = \frac{\mu}{c} V_t S_1 + \frac{1}{A}\left(W_\eta S_2 - W_\xi S_3\right),$$

where it is, of course, understood that the curl operates in the r, ξ, η coordinate system.

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12. To check the correctness of equations (5.12) one can apply the general formula (4.9) for curl to our coordinate system wherein $A_1 = 1$ and $A_2 = A_3 = A$. However, we may take advantage of the calculation made to obtain the right side of equations (5.2). For the vector whose components are E_1/A, E_2/A, E_3/A, the curl is given by the right sides of the second set of equations (5.2). For the vector $\mathcal{A}$ of (5.12), we must replace $\frac{E_1}{A}$ by $\frac{-\epsilon/c\, W_t A}{A}$, $\frac{E_2}{A}$ by $\frac{V_\eta}{A}$, and $\frac{E_3}{A}$ by $-\frac{V_\xi}{A}$. Similarly we may obtain curl $\mathcal{B}$.

By using the relations[13]

$$V_\eta = S_3 \cdot \text{grad } V$$

$$V_\xi = S_2 \cdot \text{grad } V$$

We may write

$$V_\eta S_2 - V_\xi S_3 = (S_3 \cdot \text{grad } V) S_2 - (S_2 \cdot \text{grad } V) S_3$$

$$= (S_3 \times S_2) \times \text{grad } V$$

$$= A(S_1 \times \text{grad } V)$$

in view of the fact that the $\frac{S_i}{\sqrt{A_i}}$ form an orthogonal unit set of vectors.

Hence $\mathcal{A} = -\frac{\epsilon}{c} W_t S_1 + \text{grad } V \times S_1$

Since both S_1 and grad r are unit vectors in the same direction,

$$\mathcal{A} = -\frac{\epsilon}{c} W_t \text{ grad } r + \text{grad } V \times \text{grad } r.$$

With this consideration we have expressed our main theorem independently of the coordinate system:

If V and W satisfy the equations (5.9), and if ϵ and μ are functions of r only, then a solution of Maxwell's equations is given by the expressions

$$\epsilon E = \text{curl } \mathcal{A} \quad ; \quad \mu H = \text{curl } \mathcal{B} \tag{5.13}$$

where

$$\mathcal{A} = -\frac{\epsilon}{c} W_t \text{ grad } r + \text{grad } V \times \text{grad } r$$

$$\mathcal{B} = \frac{\mu}{c} V_t \text{ grad } r + \text{grad } W \times \text{grad } r. \tag{5.14}$$

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13. See footnote 8 on the general formula for gradient.

In conclusion we express the equations (5.9) for V and W in more general form. By application of the formula for $\Delta V = \nabla^2 V$ in generalized coordinates[14] we obtain

$$\Delta V = \frac{1}{A}(V_{\xi\xi} + V_{\eta\eta}) + \frac{1}{r}(rV)_{rr}$$

Hence, from equation (5.9)

$$\frac{\epsilon\mu}{c^2} V_{tt} = \Delta V - \frac{1}{r}(rV)_{rr} + \epsilon \frac{\partial}{\partial r}\left(\frac{1}{\epsilon} V_r\right)$$

$$= \Delta V - \left(\frac{2}{r} + \frac{\epsilon'}{\epsilon}\right) V_r ,$$

and similarly

$$\frac{\epsilon\mu}{c^2} W_{tt} = \Delta W - \left(\frac{2}{r} + \frac{\mu'}{\mu}\right) W_r .$$

Thus we may formulate our main result as follows:

If V and W are solutions of the partial differential equations

$$\frac{\epsilon\mu}{c^2} V_{tt} - \Delta V = -\left(\frac{2}{r} + \frac{\epsilon'}{\epsilon}\right) V_r$$

$$\frac{\epsilon\mu}{c^2} W_{tt} - \Delta W = -\left(\frac{2}{r} + \frac{\mu'}{\mu}\right) W_r \qquad (5.15)$$

and if ϵ and μ are functions of r only, then a solution of Maxwell's equations is given by the vectors

$$\epsilon E = \operatorname{curl} \mathcal{A}; \quad \mu H = \operatorname{curl} \mathcal{B} \qquad (5.16)$$

where

$$\mathcal{A} = -\frac{\epsilon}{c} W_t \operatorname{grad} r + \operatorname{grad} V \times \operatorname{grad} r$$

$$\mathcal{B} = \frac{\mu}{c} V_t \operatorname{grad} r + \operatorname{grad} W \times \operatorname{grad} r$$

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14. See formula (82) on p. 49 of Stratton.

This result bears a remarkable analogy to the result obtained for the corresponding theory in rectangular coordinates where ϵ and μ are functions of z only[15]. This analogy is brought out by the following formalism.

Let us define a function $\varphi(x,y,z)$ by

$\varphi = z$, in the case of horizontal stratification.

and

$\varphi = r = \sqrt{x^2 + y^2 + z^2}$, in the case of spherical symmetry.

Then: If V and W satisfy the equations

$$\frac{\epsilon\mu}{c^2} V_{tt} - \Delta V = -\left(\Delta\varphi + \frac{\epsilon'}{\epsilon}\right)\frac{\partial V}{\partial\varphi} \qquad (5.17)$$

$$\frac{\epsilon\mu}{c^2} W_{tt} - \Delta W = -\left(\Delta\varphi + \frac{\mu'}{\mu}\right)\frac{\partial W}{\partial\varphi}$$

(where ϵ and μ are functions of φ alone) then Maxwell's equations are solved by

$$\epsilon E = \text{curl } \mathcal{A} \qquad (5.18)$$

$$\mu H = \text{curl } \mathcal{B}$$

where

$$\mathcal{A} = -\frac{\epsilon}{c} W_t \text{ grad } \varphi + \text{grad V x grad } \varphi$$

$$\mathcal{B} = \frac{\mu}{c} V_t \text{ grad } \varphi + \text{grad W x grad } \varphi.$$

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15. See formulas (4.7) to (4.10) in Research Report No. 172-6 referred to in footnote 2.

6. Conclusion.

The problem of ultra-high frequency propagation in a medium of spherical symmetry must now be pursued with the aid of equations (5.9) and (5.10). Several observations on these equations may be helpful. They show first of all that the problem is now a scalar one in that it reduces to the determination of two scalar functions V and W. This much could be seen from equations (3.5) also. However, the latter equations are more complex and the derivation of the other components of E and H from E_0 and H_0 would also lead to complex differential equations as compared with the simple differentiations of formulas (5.10). Of course a solution obtained in the r, ξ, η coordinate system can be reexpressed in the usual spherical coordinate system.

While the procurement of equations (5.9) and (5.10) represents a step forward it should be stressed that much more remains to be done than has been accomplished so far. Solutions of equations (5.9) which fit the conditions of the physical problem must be obtained. These conditions include the nature of the radiation source, the behavior of ϵ as a function of r, the spherical earth as a reflector, and the fact that the radiated field must behave like an outgoing wave at infinity. Given such a solution, it would then be necessary to consider the process of calculation with it. Even in the simpler, approximate treatments of the anomalous propagation problem referred to in the introduction, this phase of the problem has caused considerable difficulties.

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